JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 23, No. 2, March–April 2000

Nonlinear Six-Degree-of-Freedom Aircraft Trim

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When computing numerical trim solutions, it is difficult to know if all possible trim solutions have been found. This paper presents closed-form formulas for computing all possible trim values for six-degree-of-freedom nonlinear aircraft models. The models include gravity, nonlinear angular-rate terms, and tabular aero functions. However, the models are linear in four functions of the four inputs (thrust, aileron, elevator, rudder). The four independent control inputs lead to a four-dimensional trim set. In the subsonic case where the Mach dependence of the aero functions can be ignored, the four-dimensional trim set is parameterized by (pitch angle, roll angle, angle of attack, sideslip angle) giving at most two real trim values for each set of fixed values of these four parameters. When the Mach dependence of the aero functions needs to be considered, the four-dimensional trim set is parameterized by (speed, angle of attack, sideslip angle, and heading angular rate) giving up to eight trim values for each fixed set of values of these four parameters. The advantage of these closed-form computations, over more numerical methods, is that they give all possible solutions values.

Nomenclature

[Vectors in R^3 will be denoted by lower-case letters. All vectors in R^3 and all 3×3 matrices will be in body-axis coordinates unless specifically stated otherwise (e.g., the position vector r is with respect to an Earth-fixed coordinate frame). The transpose of a vector w is denoted by w'.]

B	= $\mathrm{diag}(b, c, b) \in R^{3\times 3}$
b	= wing span
$C^{f_h} \in R^{3\times 4}$	= tabular aero force coefficient
$C^{f_\omega} \in R^{3\times 3}$	= tabular aero force coefficient
$C^{f_0} \in R^3$	= tabular aero force coefficient
$C^{\tau_h} \in R^{3\times 4}$	= tabular aero torque coefficient
$C^{\tau_\omega} \in R^{3\times 3}$	= tabular aero torque coefficient
$C^{\tau_0} \in R^3$	= tabular aero torque coefficient
c	= mean chord length
$E \in SO(3)$	= rotation, from inertial to body axis
$e_i \in S^2$	= three columns of E $(i = 1, 2, 3)$
e_3	= $[-\sin(\theta), \cos(\theta)\sin(\phi), \cos(\theta)\cos(\phi)]'$
$f^{\text{aero,prop}} \in R^3$	= aero and propulsion forces
$G \in R^{6\times 4}$	= coefficient matrix
g	= acceleration caused by gravity
$H \in R^{6\times 4}$	= coefficient matrix for subsonic case
$H_m \in R^{6\times 7}$	= coefficient matrix
$h(x, u) \in R^4$	= four functions of (x, u)
I_2	= 2×2 identity matrix
I_3	= 3×3 identity matrix
$J_{\text{cg}} \in R^{3\times 3}$	= moment of inertia matrix
J_1, J_2, J_3	= principal moments of inertia
$K \in R^{2\times 4}$	= coefficient matrix for subsonic case
$K_m \in R^{2\times 7}$	= coefficient matrix
L	= 2×2 rotation matrix
M	= Mach number
$\hat{M} \in R^{6\times 6}$	= scaled generalized mass matrix
m	= mass of the rigid body
$\text{Poly}_i, k_{ij}, k_i$	= polynomials and coefficients
R^3	= real three-dimensional Euclidean space
$r \in R^3$	= c.g. position (Earth-fixed coordinate)
S	= wing area
$SO(3)$	= 3×3 real orthogonal matrices, det = +1
S^2	= space of all unit vectors in R^3
T	= thrust
u	= $(T, \delta_a, \delta_e, \delta_r)'$ thrust, aileron, elevator, rudder
$v \in R^3$	= velocity of the c.g. of the rigid body
v_c	= $\sqrt{[mg/\frac{1}{2}\rho S]}$, constant, dimensions of speed [Note: $(\|v\|/v_c)^2 = \frac{1}{2}\rho\|v\|^2 S/mg$]
x	= $\begin{pmatrix} x_{\text{dyn}} \\ x_{\text{kin}} \end{pmatrix}$
x_{dyn}	= $\begin{pmatrix} v \\ \omega \end{pmatrix} \in R^6$ = dynamic states
x_{kin}	= $(r', e_1', e_2', e_3')'$ with constraints $E'E = I_3$
$[x, y, z]$	= e_3' (only used in Secs. VII and VIII)
ρ	= density of air
$\tau^{\text{aero,prop}} \in R^3$	= aero and propulsion moments
ϕ, θ, ψ	= Euler angles
$\dot{\psi}$	= $\omega' e_3$ = rate of change of heading angle
$\omega \in R^3$	= angular rate of the rigid body

Received 16 July 1998; revision received 9 September 1999; accepted for publication 10 September 1999.

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I. Introduction

REFERENCE 1 gives an algebraic solution for aircraft trim when the aero functions are linear functions of angle of attack and sideslip angle, but gravitational force is ignored. Reference 2 gives a numerical procedure for computing aircraft trim and stability for general nonlinear aircraft models. The current paper is based on previous results from Refs. 3–5, where a closed-form solution was given for all trim values when the aero functions are arbitrary functions of angle of attack and sideslip angle.

The air density and aircraft mass will both be assumed constant, and effects caused by a spherical rotating Earth will be ignored. Two cases are considered. In the subsonic case Mach effects are ignored resulting in simple trim equations. In the second case the aero functions are allowed to be tabular in Mach as well as in angle of attack and sideslip angle.

The set of points in the (state, control) space that satisfy the trim equations depend only on the properties of the aircraft itself and not on the control algorithm. However, a control algorithm does determine the dynamic stability of the points on the trim set. This paper deals only with computing the trim set, not with stability or control issues.

Section II summarizes the equations of motion. Section III gives the general trim equations. Section IV presents the subsonic trim equations, and Sec. V gives their solution. Section VI presents a subsonic numerical example. Section VII gives the Mach-dependent trim equations, and Sec. VIII gives their solution. Section IX shows a transonic example, and Sec. X summarizes the conclusions.

II. Aircraft Equations of Motion

Details on the aircraft equations of motion can be found in Ref. 6. In this paper the aero and thrust effects are approximated as affine in ω and $h(x,u)$ with coefficients that are table look up in angle of attack and sideslip angle (i.e., in the unit vector $v/\|v\|$). These coefficients are also tabular in the Mach number in Secs. VII–IX.

$$(1/mg)f^{\text{aero,prop}} = (\|v\|/v_c)^2[C^{f_0} + C^{f_\omega}(B\omega/2\|v\|)] + C^{f_h}h(x,u) \tag{1}$$

$$(mgB)^{-1}\tau^{\text{aero,prop}} = (\|v\|/v_c)^2[C^{\tau_0} + C^{\tau_\omega}(B\omega/2\|v\|)] + C^{\tau_h}h(x,u) \tag{2}$$

The function $h(x,u) \in R^4$ could be any nonlinear function of (x,u). The simplest example is

$$h(x,u) = \begin{bmatrix} T/mg \\ \left(\frac{\|v\|}{v_c}\right)^2 \begin{pmatrix} \delta_a \\ \delta_e \\ \delta_r \end{pmatrix} \end{bmatrix} \tag{3}$$

The most complicated example this model allows would be a general tabular function $h(x,u)$.

In body-axis coordinates the dynamics can be written as

$$\begin{bmatrix} \dot{v}/g \\ (B^{-1}/mg)J_{\text{cg}}\dot{\omega} \end{bmatrix} = \begin{bmatrix} -(\omega\times v)/g + e_3 \\ -(B^{-1}/mg)\omega\times J_{\text{cg}}\omega \end{bmatrix} + \left(\frac{\|v\|}{v_c}\right)^2 \left(\begin{bmatrix} C^{f_0} \\ C^{\tau_0} \end{bmatrix} + \begin{bmatrix} C^{f_\omega} \\ C^{\tau_\omega} \end{bmatrix}\frac{B\omega}{2\|v\|}\right) + \begin{bmatrix} C^{f_h} \\ C^{\tau_h} \end{bmatrix} h \tag{4}$$

The kinematics are given by

$$\dot{r} = E'v, \qquad \dot{e}_i = -\omega\times e_i \qquad i = 1,2,3 \tag{5}$$

Equations (4) and (5) are of the form

$$\hat{M}\dot{x}_{\text{dyn}} = f_{\text{dyn}}(x_{\text{dyn}}, x_{\text{kin}}, u) \tag{6}$$

$$\dot{x}_{\text{kin}} = f_{\text{kin}}(x_{\text{dyn}}, x_{\text{kin}}) \tag{7}$$

However, f_{dyn} only depends on the e_3 part of x_{kin}.

III. General Trim Equations

This section gives the trim equations without specifying a solution technique. Later sections will give a simple solution technique for the subsonic case and a somewhat more complicated solution technique for the case where the aero coefficients have significant Mach variations.

First a distinction is made between the words trim and equilibrium. Equilibria of a dynamical system of the form given in the preceding section are solutions of the equations $0 = f_{\text{dyn}}(x_{\text{dyn}}, x_{\text{kin}}, u)$ and $0 = f_{\text{kin}}(x_{\text{dyn}}, x_{\text{kin}})$. Trim of a dynamical system of the form given in the preceding section are solutions of the equations $0 = f_{\text{dyn}}(x_{\text{dyn}}, x_{\text{kin}}, u)$ with u constant.

If equilibria were computed, then $f_{\text{kin}} = 0$ would imply that r was constant, leaving hovering flight as the only equilibria. However, if only $f_{\text{dyn}} = 0$, then the solutions to those (trim) equations will be vertical helices in the (v, ω, r, E) state space.[6] Vertical helices include straight-line flight, horizontal circular trajectories, and hovering flight as special cases.

In the case of an aircraft, f_{dyn} depends on all elements of x_{dyn} but on just some of the elements of x_{kin}. In particular, $x_{\text{kin}} = [r', e_1', e_2', e_3']'$, while f_{dyn} only depends on (v, ω, e_3) and the input vector u. So solving the trim equations amounts to solving

$$f_{\text{dyn}}(v, \omega, e_3, u) = 0 \in R^6 \qquad u \text{ const} \tag{8}$$

Because $f_{\text{dyn}} = 0$, $x_{\text{dyn}} = (v, \omega)$ is constant. Because f_{dyn} depends on e_3 in a full rank way, e_3 must also be constant. Because $0 = \dot{e}_3 = -\omega\times e_3$, this implies that at trim $\omega = (\omega' e_3)e_3 = \dot{\psi}e_3$.

Using $\omega = \dot{\psi}e_3$ in Eq. (8) gives the trim equations as

$$f_{\text{dyn}}(v, \dot{\psi}e_3, e_3, u) = 0 \in R^6 \qquad u \text{ const} \tag{9}$$

The assumption that f_{dyn} is affine in some vector function $h(x,u) \in R^4$, with 6×4 coefficient matrix G that depends only on v, means that the trim equations reduce to

$$f_{\text{dyn}} = F(v, \dot{\psi}, e_3) + G(v)h(x,u) = 0 \in R^6 \tag{10}$$

Setting $\dot{x}_{\text{dyn}} = 0$ in Eq. (4) gives the coefficient matrices F and G in Eq. (10):

$$F = \begin{bmatrix} (\dot{\psi}\|v\|/g)(v/\|v\|\times e_3) + e_3 \\ (\dot{\psi}^2 B^{-1}/mg)[(J_{\text{cg}}e_3)\times e_3] \end{bmatrix} + \left(\frac{\|v\|}{v_c}\right)^2\left(\begin{bmatrix} C^{f_0} \\ C^{\tau_0} \end{bmatrix} + \frac{\dot{\psi}}{\|v\|}\begin{bmatrix} C^{f_\omega} \\ C^{\tau_\omega} \end{bmatrix}\frac{Be_3}{2}\right) \tag{11}$$

and

$$G = \begin{bmatrix} C^{f_h} \\ C^{\tau_h} \end{bmatrix} \tag{12}$$

If $\text{rank}[G] = 4$, then for each setting of the four controls the equations of motion will have corresponding trim points for the state variables. Therefore the trim set is a four-dimensional subset of the state and control space. Instead of fixing the four values of the controls then solving for the trim values of the states,[1,2] the derivation below holds four of the states fixed then determines the trim values of the remaining states and controls. This results in simpler equations.

In the next section the subsonic case is considered, where Mach dependence can be ignored. In Secs. VII and VIII the aero coefficients will be allowed to depend on Mach.

IV. Subsonic Trim Equations

In the subsonic case Mach effects can be ignored so that the trim equations become quadratic in $[\|v\|, \dot{\psi}, e_3]$, linear in h, and tabular in $v/\|v\|$.

In this section four free parameters in the unit vectors $v/\|v\|$ and e_3 will be used to parameterize the trim set, then the trim equations for $\|v\|$, $\dot{\psi}$ will be solved and then the four components of h will be computed. If $h(x,u)$ is invertible with respect to u, the thrust setting, aileron angle, elevator angle, and rudder angle can also be computed.

Explicitly separating out the $\|v\|$ and $\dot{\psi}$ dependence in Eq. (11) gives

$$F = H\left(\frac{v}{\|v\|}, e_3\right)\begin{bmatrix} 1 \\ (\|v\|/v_c)^2 \\ (\dot{\psi}v_c/g)(\|v\|/v_c) \\ (\dot{\psi}v_c/g)^2 \end{bmatrix} \tag{13}$$

where $H[(v/\|v\|), e_3] \in R^{6\times 4}$ is given by

$$\begin{bmatrix} e_3 & C^{f_0} & \left(\frac{v}{\|v\|}\times e_3\right) + C^{f_\omega}\frac{gBe_3}{2v_c^2} & 0 \\ 0 & C^{\tau_0} & C^{\tau_\omega}\frac{gBe_3}{2v_c^2} & \frac{gB^{-1}[(J_{\text{cg}}e_3)\times e_3]}{mv_c^2} \end{bmatrix} \tag{14}$$

and G is the following 6×4 matrix:

$$G\left(\frac{v}{\|v\|}\right) = \begin{bmatrix} C^{f_h}(v/\|v\|) \\ C^{\tau_h}(v/\|v\|) \end{bmatrix} \tag{15}$$

The computation proceeds as follows. Because G is generically rank 4, it is possible to construct a 6×2 orthogonal complement $G^\perp$, such that $(G^\perp)'G = 0_{2\times 4}$ and $(G^\perp)'G^\perp = I_2$. The only freedom in choosing $G^\perp$ is that it can be replaced by $G^\perp L$, where L is any 2×2 rotation matrix: $LL' = L'L = I_2$.

Let $K(v/\|v\|, e_3)$ be the following 2×4 matrix:

$$K = (G^\perp)'H \tag{16}$$

Multiplying Eq. (10) by $(G^{\perp})'$ and using Eqs. (13) and (16) gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = K\left(\frac{v}{\|v\|}, e_3\right)\begin{bmatrix} 1 \\ (\|v\|/v_c)^2 \\ (\dot{\psi}v_c/g)(\|v\|/v_c) \\ (\dot{\psi}v_c/g)^2 \end{bmatrix} \quad (17)$$

Note that if the aircraft (including actuators, body shape, and mass properties) has a plane of symmetry and both e_3 and $v/\|v\|$ lie in that plane of symmetry, then column 3 of K is orthogonal to columns 1, 2, and 4 of K.

Equation (17) represents the subsonic trim equations. The coefficient matrix K is a function of the four parameters in the two unit vectors $v/\|v\|$ and e_3. For each fixed value of those four parameters, the resulting K matrix determines the corresponding points on the four-dimensional trim set. The next section gives the solutions to Eq. (17) for each fixed value of K.

V. Solution of the Subsonic Trim Equations

If the 2×4 matrix $K(v/\|v\|, e_3)$ in Eq. (17) has rank less than 2, then the two scalar equations in Eq. (17) are redundant. In that case there are either no solutions or an infinite number of $(\|v\|, \dot{\psi})$ solutions for that set of $(v/\|v\|, e_3)$ values. Next, the generic case where $\mathrm{rank}[K(v/\|v\|, e_3)] = 2$ is considered.

When $K \in R^{2\times 4}$ is rank 2, then it has a two-dimensional right kernel. The right kernel of K is given by $A \in R^{4\times 2}$, where A is the solution matrix of the equation:

$$KA = 0 \in R^{2\times 2} \quad (18)$$

Any vector $n \in R^4$ will satisfy $Kn = 0$ if n is a linear combination of the two columns of the matrix A:

$$n = As \qquad s \in R^2 \quad (19)$$

Equation (17) shows a vector in the kernel of K so that it must be of the preceding form.

$$\begin{bmatrix} 1 \\ (\|v\|/v_c)^2 \\ (\dot{\psi}v_c/g)(\|v\|/v_c) \\ (\dot{\psi}v_c/g)^2 \end{bmatrix} = As \qquad s \in R^2 \quad (20)$$

The only unknowns in the preceding equation are the two elements of $s \in R^2$. Let a_i denote row i of A. The top row of Eq. (20) gives one scalar equation for $s \in R^2$.

$$1 = a_1 s \quad (21)$$

A second equation can be obtained by multiplying the second and fourth row of Eq. (20), then subtracting the square of row three. The result is $0 = (\|v\|/v_c)^2(\dot{\psi}v_c/g)^2 - [(\|v\|/v_c)(\dot{\psi}v_c/g)]^2$, or

$$0 = (a_2 s)(a_4 s) - (a_3 s)^2 = s'[a_2' a_4 - a_3' a_3]s \quad (22)$$

The symmetric part of the matrix in the preceding equation is $D = ([a_2' a_4 - a_3' a_3] + [a_2' a_4 - a_3' a_3]')/2$, so Eq. (22) can be rewritten as $0 = s' Ds$.

Let λ_1 and λ_2 be the real eigenvalues of the symmetric D matrix, ordered so that $\lambda_1 \geq \lambda_2$. Let w_1 and w_2 be the corresponding orthogonal eigenvectors. Then $0 = s' Ds$ becomes

$$0 = \left[(\sqrt{\lambda_1}w_1 + \sqrt{-\lambda_2}w_2)'s\right]\left[(\sqrt{\lambda_1}w_1 - \sqrt{-\lambda_2}w_2)'s\right] \quad (23)$$

Therefore the two solutions for the vector $s \in R^2$ from Eq. (19) are obtained by simultaneously solving Eqs. (21) and (23).

$$\begin{bmatrix} a_1 \\ (\sqrt{\lambda_1}w_1 \pm \sqrt{-\lambda_2}w_2)' \end{bmatrix} s = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (24)$$

Because $\|v\|/v_c \geq 0$, row 2 of Eq. (20) can then be used to determine a unique value for $\|v\|/v_c$ for each of the two values of the vector s.

$$\|v\|/v_c = \sqrt{a_2 s} \quad (25)$$

Then row 3 of Eq. (20) can be used to obtain a unique value of $\dot{\psi}v_c/g$ for each value of $\|v\|/v_c$.

$$\dot{\psi}v_c/g = (a_3 s)/(\|v\|/v_c) \quad (26)$$

Numerical evaluation of several aircraft models show that at small angles of attack typically one of the two $\|v\|/v_c$ solutions is larger than 1, while the other is either much smaller than 1 or complex. The equilibria associated with large speeds are typically associated with low $\dot{\psi}v_c/g$ values and correspond to reasonable flight regimes, whereas the low-speed equilibria are often associated with large $\dot{\psi}v_c/g$, spin-type flight. Very low speeds result in the aerodynamic control surfaces not having enough authority to hold the aircraft at that equilibrium (input saturation). If the aileron, elevator, and rudder are replaced by thrust vectoring, then the $v = 0$ solutions are allowed.

Given the values of $\|v\|/v_c$ and $\dot{\psi}v_c/g$, Eqs. (10) and (13) can be used to solve for $h(x, u)$ using linear algebra.

$$h(x,u) = -(G'G)^{-1}G'H\begin{bmatrix} 1 \\ (\|v\|/v_c)^2 \\ (\dot{\psi}v_c/g)(\|v\|/v_c) \\ (\dot{\psi}v_c/g)^2 \end{bmatrix} \quad (27)$$

Finally, use this value of $h(x, u)$, together with the calculated value of $\|v\|/v_c$, to solve for the four inputs $[T/(mg), \delta_a, \delta_e, \delta_r]$ using Eq. (3).

VI. Subsonic Example

The following numerical example is used to illustrate the derivations of earlier sections. The aero data could have been tabular in angle of attack and sideslip angle (i.e., functions of $v/\|v\|$), but to keep the example simple, trig functions and constants were used to give aero data representative of modern fighter aircraft.

The mass properties and geometry are given by

$$J_{cg} = \begin{bmatrix} 46{,}477 & 81.3 & 13.6 \\ 81.3 & 257{,}605 & 651 \\ 13.6 & 651 & 297{,}602 \end{bmatrix} \mathrm{kg\ m^2} \quad (28)$$

$$m = 19{,}483\ \mathrm{kg}, \qquad g = 9.81\ \mathrm{m/s^2}, \qquad S = 52.5\ \mathrm{m^2} \quad (29)$$

$$\rho = 1.22\ \mathrm{kg/m^3}, \qquad b = 19.54\ \mathrm{m}, \qquad c = 2.99\ \mathrm{m} \quad (30)$$

$$B = \mathrm{diag}([b, c, b]), \qquad v_c = \sqrt{mg/\tfrac{1}{2}\rho S} \quad (31)$$

The 3×1 aero force and moment vectors were given by

$$C^{f0}\left(\frac{v}{\|v\|}\right) = \begin{bmatrix} -0.05\cos(\alpha)\cos(\beta) \\ -0.76\sin(\beta) \\ -4.3\sin(\alpha)\cos(\beta) \end{bmatrix} \quad (32)$$

$$C^{\tau 0}\left(\frac{v}{\|v\|}\right) = \begin{bmatrix} -0.13\sin(\beta) \\ -0.73\sin(\alpha)\cos(\beta) \\ 0.27\sin(\beta) \end{bmatrix} \quad (33)$$

The 3 × 3 aero coefficient matrices that multiply the angular-rate vector where given constant values are as follows:

$$C^{f\omega}\left(\frac{v}{\|v\|}\right) = \begin{bmatrix} 0 & 0 & 0 \\ -0.044 & 0 & 2.3 \\ 0 & -73 & 0 \end{bmatrix} \tag{34}$$

$$C^{\tau\omega}\left(\frac{v}{\|v\|}\right) = \begin{bmatrix} -1.4 & 0 & 0.55 \\ 0 & -42 & 0 \\ -0.022 & 0 & -1.2 \end{bmatrix} \tag{35}$$

The 3 × 4 aero coefficient matrices that multiply the scaled input vector $h(x, u)$ were also constant.

$$C^{f_h}\left(\frac{v}{\|v\|}\right) = \begin{bmatrix} 0.99 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.08 \\ 0.09 & 0 & 0.08 & 0 \end{bmatrix} \tag{36}$$

$$C^{\tau_h}\left(\frac{v}{\|v\|}\right) = \begin{bmatrix} 0 & 0.06 & 0 & 0.007 \\ 0.4 & 0 & 0.36 & 0 \\ 0 & 0.17 & 0 & -0.06 \end{bmatrix} \tag{37}$$

In Fig. 1 $\beta = 0.1$ rad, $\phi = 0.2$ rad, and $\theta = 0.3$ rad. The horizontal axis has $0 \leq \alpha \leq 0.3$ rad. The vertical axis has $\|v\|/v_c$ plotted with × and $\dot{\psi} v_c/g$ plotted with circles. As expected, speed decreases as angle of attack increases. Figure 2 shows the corresponding values of the inputs: $h(x, u) = [\text{Thrust}/mg, (\delta_a, \delta_e, \delta_r) * (\|v\|/v_c)^2]$. Because the surface deflections plots are scaled by speed squared, the plots decrease with increasing α, even though the surface deflections themselves are increasing with α.

VII. Mach-Dependent Case

In high-speed flight regimes the Mach dependence of the aero functions cannot be ignored. In this case the four-dimensional trim set can be parameterized using speed, angle of attack, sideslip angle, and rate of change of heading angle. This allows the aero functions to be tabular in those four variables. Given any set of four values for these four variables, the trim equations can be solved for e_3 and the four control inputs $u = [\text{Thrust}, \delta_a, \delta_e, \delta_r]$.

The trim equations have been assumed to be affine in $h(x, u)$. The trim equations depend on e_3 in a quadratic way, which can be seen as follows. The gravitational force is linear in e_3. At trim $\omega = (\omega' e_3)e_3 = \dot{\psi} e_3$ so that the $v \times \omega$ term is linear in e_3. The only remaining dependence of f_{dyn} on e_3 is from the $\omega \times J\omega$ term. Let the three components of e_3 be denoted by $[x, y, z]$. Then in principal body axis J is diagonal, and at trim

$$\omega \times J\omega = \dot{\psi}^2 \begin{bmatrix} (J_3 - J_2)yz \\ (J_1 - J_3)xz \\ (J_2 - J_1)xy \end{bmatrix} \tag{38}$$

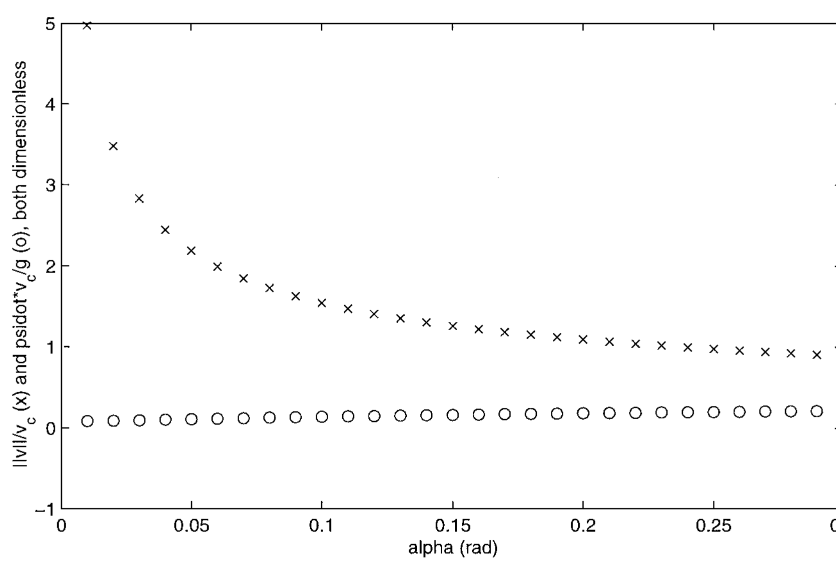

Fig. 1 $\|v\|/v_c$ **and** $\dot{\psi} v_c/g$ **when** $(\beta, \phi, \theta) = (0.1, 0.2, 0.3)$ **rad.**

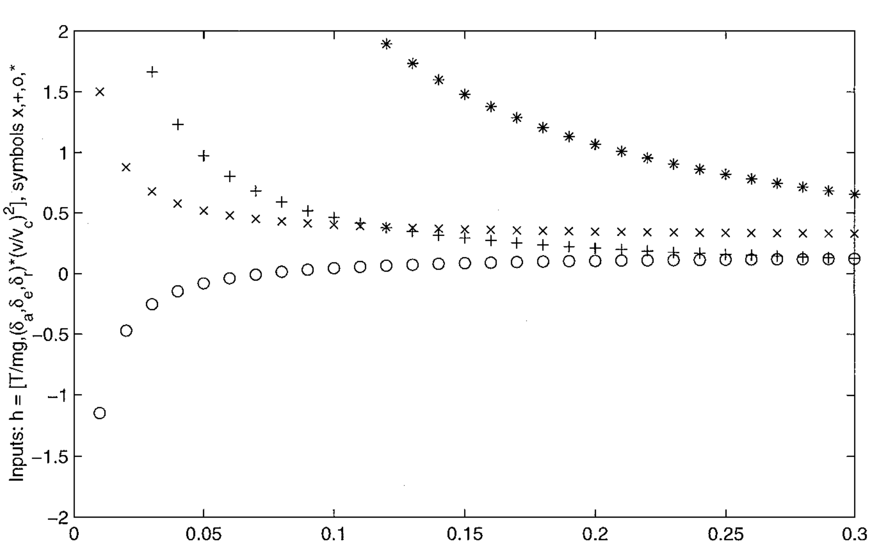

Fig. 2 Inputs when $(\beta, \phi, \theta) = (0.1, 0.2, 0.3)$ **rad.**

The trim equations (10) then are of the form:

$$H_m(v,\dot\psi)\begin{bmatrix}1\\x\\y\\z\\yz\\xz\\xy\end{bmatrix}+G(v)h(x,u)=0\in R^6 \tag{39}$$

If the preceding equation is multiplied by the 2×6 matrix $[G(v)]^\perp$ and the 2×7 matrix K_m is defined by

$$K_m(v,\dot\psi)=[G(v)]^\perp H_m(v,\dot\psi) \tag{40}$$

then the preceding equation gives

$$K_m(v,\dot\psi)\begin{bmatrix}1\\x\\y\\z\\yz\\xz\\xy\end{bmatrix}=0\in R^2 \tag{41}$$

Combining this with a third equation

$$0=x^2+y^2+z^2-1 \tag{42}$$

gives three quadratic equations in the components of the unit vector $e_3=[x,y,z]$.

Let K_{mi} be the ith column of K_m. When rank($[K_{m5},K_{m6},K_{m7}]$) = 2, these three equations are each degree 2 so that by Bezout's theorem[7] there are $2^3=8$ solutions.

VIII. Solving the Mach-Dependent Trim Equations

When the last three columns of the 2×7 matrix K_m are rank 2, there are eight solutions to the three quadratic equations. Before looking at that generic case, consider the special case where the last three columns of the 2×7 matrix $K_m(v,\dot\psi)$ are all zero or very small. This occurs when $\dot\psi=0$ or is very small.

When the last three columns of $K_m(v,\dot\psi)$ are negligibly small, the remaining four columns give $0=[K_{m1},K_{m2},K_{m3},K_{m4}]*[1,x,y,z]'$, which are linear. Combining these two linear equations with the remaining quadratic equation $1=x^2+y^2+z^2$ gives two solutions. Therefore in flight conditions (specified by $v\in R^3$ and $\dot\psi$) where $\dot\psi$ is very small, there will be only two solutions for e_3 and the control inputs.

To solve for the two solutions in the small $\dot\psi$ case, let the singular value decomposition of the $[K_{m2},K_{m3},K_{m4}]$ matrix be $U[\Sigma,0]V'$, where $U\in R^{2\times 2}$, $V\in R^{3\times 3}$, and $\Sigma\in R^{2\times 2}$ is diagonal with $\sigma_1([K_{m2},K_{m3},K_{m4}])$ and $\sigma_2([K_{m2},K_{m3},K_{m4}])$ on the diagonal. Then the two solutions are

$$e_3=\begin{bmatrix}x\\y\\z\end{bmatrix}=V\begin{bmatrix}-\Sigma^{-1}U'K_{m1}\\ \pm\sqrt{1-\|\Sigma^{-1}U'K_{m1}\|^2}\end{bmatrix} \tag{43}$$

In the unusual case where the 2×3 submatrix $[K_{m5},K_{m6},K_{m7}]$ has rank 1, the system of three quadratic equation reduces to two quadratic and one linear equation, with four solutions.

When $\dot\psi$ is not very small, it is necessary to consider the general case using all columns of the rank 2 matrix K_m. The eight solutions for the case where rank ($[K_{m5},K_{m6},K_{m7}]$) = 2 are computed as follows. Some pair of two columns of $[K_{m5},K_{m6},K_{m7}]$ will form a 2×2 invertible matrix. After possible reordering of the $[x,y,z]$ components, the last two columns of K_m form a 2×2 invertible matrix. In that case multiplying K_m by the inverse of that 2×2 matrix gives the following 2×5 matrix:

$$\begin{bmatrix}a_1&f_1&b_1&c_1&d_1\\a_2&f_2&b_2&c_2&d_2\end{bmatrix}=[K_{m6},K_{m7}]^{-1}[K_{m1},\ldots,K_{m5}] \tag{44}$$

Let $m_i(y,z)=a_i+b_iy+c_iz+d_iyz$. Let $[K_{m6},K_{m7}]^{-1}\times$ Eq. (41) define $[p_1,p_2]'$. Let p_3 be the polynomial in Eq. (42). Then

$$\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}=\begin{bmatrix}m_1(y,z)&f_1+z&0\\m_2(y,z)&f_2+y&0\\y^2+z^2-1&0&1\end{bmatrix}\begin{bmatrix}1\\x\\x^2\end{bmatrix} \tag{45}$$

Because polynomials p_1 and p_2 do not have x^2 terms, they can be multiplied by x to obtain two more equations, for a total of five equations of the preceding form.

Let

$$R_m(y,z)=\begin{bmatrix}m_1(y,z)&f_1+z&0\\m_2(y,z)&f_2+y&0\\y^2+z^2-1&0&1\\0&m_2(y,z)&f_2+y\\0&m_1(y,z)&f_1+z\end{bmatrix} \tag{46}$$

then

$$\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}=\begin{bmatrix}p_1\\p_2\\p_3\\xp_2\\xp_1\end{bmatrix}=R_m(y,z)\begin{bmatrix}1\\x\\x^2\end{bmatrix} \tag{47}$$

Because the 5×3 R_m matrix multiplies a nonzero vector and the result is zero, R_m must have rank 2 or less for solutions to exist. Let R_{mi} be row i of R_m. The matrix R_m will be rank 2 or less if $0=\det([R_{m1},R_{m2},R_{m3}])$ and $\det([R_{m2},R_{m3},R_{m4}])$, except if any of the common rows (row 2 or 3) are identically zero. Row 3 cannot be identically zero. Row 2 can be identically zero if $y=-f_2$ and z is a solution of $0=m_2(-f_2,z)$. By changing coordinates, row 2 can be prevented from ever being zero. Let $w=1/(f_2+y)$. Then $y=(1-f_2w)/w$. Substituting this into $R_m(y,z)$ and multiplying rows 1, 2, 4, 5 by w and row 3 by w^2 gives a polynomial matrix $\widehat{R_m}(w,z)$. Let $r_i(w,z)=(a_i+c_iz)w+(b_i+d_iz)(1-f_2w)$, then

$$\widehat{R_m}(w,z)=\operatorname{diag}([w,w,w^2,w,w])\,R_m[(1-f_2w)/w,z]$$
$$=\begin{bmatrix}r_1(w,z)&f_1+z&0\\r_2(w,z)&1&0\\(1-f_2w)^2+(z^2-1)w^2&0&w^2\\0&r_2(w,z)&1\\0&r_1(w,z)&f_1+z\end{bmatrix} \tag{48}$$

Taking the determinants of two 3×3 submatrices of $\widehat{R_m}$ gives two equations in w, z, which can then be used to form a single equation in w alone. The determinants of these 3×3 matrices are called resultants.[7]

Let

$$S_0(w)=wa_1+(1-f_2*w)b_1-wf_1[wa_2+(1-f_2w)b_2] \tag{49}$$

$$S_1(w)=(-a_2-c_2f_1+b_2f_2+d_2f_1f_2)w^2+(-b_2+c_1-d_2f_1-d_1f_2)w+d_1 \tag{50}$$

$$S_2(w)=-wc_2-(1-f_2w)d_2 \tag{51}$$

$$T_0(w)=(1-f_2w)^2-w^2+\{w[wa_2+(1-f_2w)b_2]\}^2 \tag{52}$$

$$T_1(w)=2[wa_2+(1-f_2w)b_2][wc_2+(1-f_2w)d_2] \tag{53}$$

$$T_2(w)=1+[wc_2+(1-f_2w)d_2]^2 \tag{54}$$

Let $\widehat{R_{m_i}}$ denote row i of $\widehat{R_m}$. The determinants of two 3×3 submatrices of $\widehat{R_m}$ can then be combined into another matrix equation as follows:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \det([\widehat{R_{m1}}, \widehat{R_{m2}}, \widehat{R_{m3}}]) \\ w\, z \det([\widehat{R_{m1}}, \widehat{R_{m2}}, \widehat{R_{m3}}]) \\ \det([\widehat{R_{m2}}, \widehat{R_{m3}}, \widehat{R_{m4}}]) \\ z \det([\widehat{R_{m2}}, \widehat{R_{m3}}, \widehat{R_{m4}}]) \end{bmatrix}$$

$$= \begin{bmatrix} S_0(w) & S_1(w) & S_2(w) & 0 \\ 0 & w S_0(w) & S_1(w) & S_2(w) \\ T_0(w) & w^2 T_1(w) & w T_2(w) & 0 \\ 0 & T_0(w) & w T_1(w) & T_2(w) \end{bmatrix} \begin{bmatrix} 1 \\ z \\ w z^2 \\ w^2 z^3 \end{bmatrix} \tag{55}$$

Let

$$Q(w) = \begin{bmatrix} S_0(w) & S_1(w) & S_2(w) & 0 \\ 0 & w S_0(w) & S_1(w) & S_2(w) \\ T_0(w) & w^2 T_1(w) & w T_2(w) & 0 \\ 0 & T_0(w) & w T_1(w) & T_2(w) \end{bmatrix} \tag{56}$$

The 4×4 matrix $Q(w)$ multiplies a nonzero vector, yet the result is zero, therefore the determinant of $Q(w)$ must be zero.

$$0 = \det[Q(w)] = S_2^2 T_0^2 + S_1^2 T_0 T_2 - S_2 S_1 T_0 T_1 w - 2 S_2 S_0 T_0 T_2 w - S_0 S_1 T_1 T_2 w^2 + S_0^2 T_2^2 w^2 + S_2 S_0 T_1^2 w^3 \tag{57}$$

Each of the seven terms in the preceding expression, e.g., $S_2^2 T_0^2$, is a degree 10 polynomial in w. However, when the seven terms are added together, the result is a degree 8 polynomial in w because the coefficients on w^9 and w^{10} are both zero. This is caused by the change of coordinates $w = 1/(f_2 + y)$.

Let $w_i, i = 1, 2, \ldots, 8$ be the eight roots of the polynomial $\det[Q(w)]$. For each of these eight values, the $Q(w)$ matrix will be singular, with right kernel $[1, z, wz^2, w^2 z^3]'$. The eight corresponding values of $z_i, i = 1, 2, \ldots, 8$ can be obtained by numerically computing the right kernel vec_i such that $0 = Q(w_i)\mathrm{vec}_i$. Then $z_i = \mathrm{vec}_i(2)/\mathrm{vec}_i(1)$. The eight corresponding values of x are obtained by $x_i = -w_i(a_2 + c_2 z_i) - (1 - f_2 w_i)(b_2 + d_2 z_i)$. Using the change of coordinates from w to y gives the eight corresponding values $y_i = (1 - f_2 w_i)/w_i$. These eight values of $e_{3i} = [x_i, y_i, z_i]'$ then form the eight solutions to the trim problem. If a constant rotation was done to diagonalize the moment of inertia matrix, then that rotation must be applied to e_3 to put e_3 back into the original coordinate system. Given e_3 as well as the specified values of $(\|v\|, \alpha, \beta, \dot{\psi})$, the original trim equations can then be solved for the inputs u.

IX. Transonic Example

The preceding section gave a method for solving the trim equations when the aero functions are tabular in $v/\|v\|$ and M, i.e., tabular in α, β, and M. To illustrate the technique, this section uses a simple analytical example based on the subsonic example from an

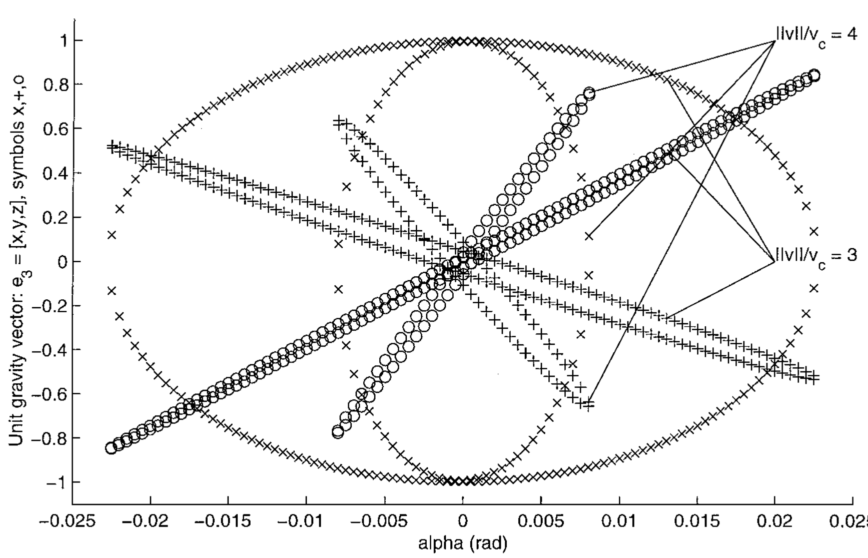

Fig. 3 e_3 when speeds/v_c = 3 and 4, $\dot{\psi} v_c/g = -0.2$, and $\beta = .005$ rad.

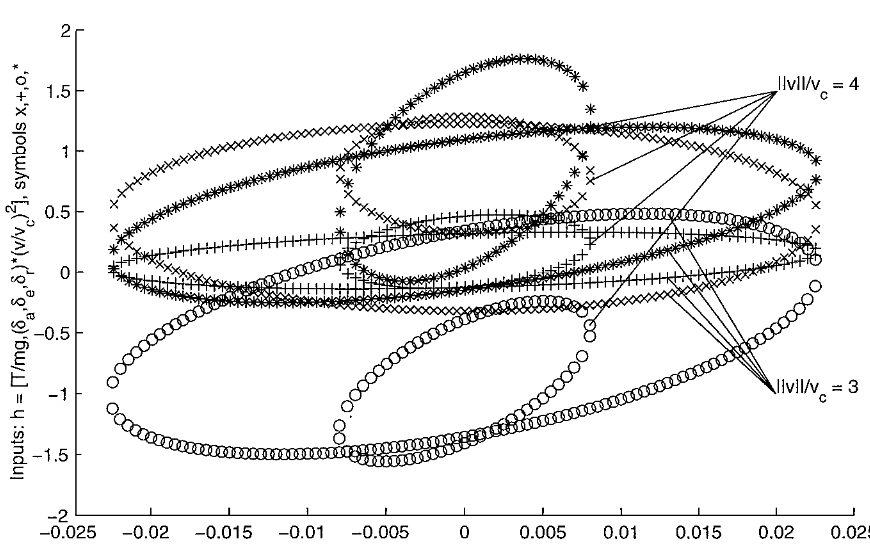

Fig. 4 Inputs when speeds/v_c = 3 and 4, $\dot{\psi} v_c/g = -0.2$, and $\beta = 0.005$ rad.

earlier section. To include Mach dependence, all aero functions are modified in the following way:

$$C\left(\frac{v}{\|v\|}, M\right) = \frac{1.05 C_{\text{subsonic}}(v/\|v\|)}{\sqrt{(1+M)[0.1+|1-M|^3/(0.01+|1-M|^2)]}} \tag{58}$$

For $|1 - M| \gg 0.1$ the formula is approximately $C(v/\|v\|, M) \approx C_{\text{subsonic}}(v/\|v\|)/\sqrt{[(1+M)|1-M|]}$. The example used $M = \|v\|/(335 \text{ m/s})$.

Figure 3 shows the trim values of $e_3 = [x, y, z]$ vs α at $\beta = 0.005$ rad, $\|v\|/v_c = 3$ or 4, and $\dot{\psi} v_c/g = -0.2$. At each value of α, there were eight sets of solutions for $[x, y, z]$; however, at most two of the sets were real in this example. Figure 4 shows the corresponding values of the dimensionless input vector $[T/mg, (\delta_a, \delta_e, \delta_r) * (v/v_c)^2]$.

The same model can be used to construct a low-speed, high angular-rate example where all eight solutions are real. This occurs when

$$\dot{\psi} v_c/g = 40 \quad (\dot{\psi} = 5.08 \text{ rad/s}), \qquad \beta = 0 \text{ rad}$$

$$\|v\|/v_c = 0.001 \quad (\|v\| = 0.25 \text{ ft/s}), \qquad \alpha = 1.4 \text{ rad}$$

Only one of the resulting eight solutions has reasonable values of thrust and surface deflections.

X. Conclusions

The trim solutions described in this paper give a closed-form expression for computing all possible trim solutions of the nonlinear aircraft model given. The model allows aero functions that are arbitrary functions of angle of attack and sideslip angle (and Mach number, for the second solution type).

The most serious limitation of the model is that it is linear in the thrust and surface deflections. This means that for fixed values of the state, varying the controls can only affect a four-dimensional linear subset of the six-dimensional (force, torque) space. This is a typical assumption in most nonlinear control theory, but more realistic aircraft models would allow more general nonlinear dependence on the inputs. Actuator saturation would also prevent solving for the surface deflections.

The description given of the entire four-dimensional trim set could be of use in further studies of the stability of the six-degree-of-freedom nonlinear aircraft equations of motion.

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